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Wave functionals, gauge invariant equations for massive modes and the Born-Infeld equation in the loop variable approach to string theory

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Abstract: In earlier papers on the loop variable approach to gauge invariant interactions in string theory, a "wave functional" with some specific properties was invoked. It had the purpose of converting the generalized momenta to space time fields. In this paper we describe this object in detail and give some explicit examples. We also work out the interacting equations of the massive mode of the bosonic string, interacting with electromagnetism, and discuss in detail the gauge invariance. This is naturally described in this approach as a massless spin two field interacting with a massless spin one field in a higher dimension. Dimensional reduction gives the massive system. We also show that in addition to describing fields perturbatively, as is required for reproducing the perturbative equations, the wave functional can be chosen to reproduce the Born-Infeld equations, which are non-perturbative in field strengths. This makes contact with the sigma model approach.

Keywords: Bosonic Strings, Gauge Symmetry, Conformal and W Symmetry.

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Contents

1. Introduction

One of the outstanding issues in string theory is that of understanding the fundamental symmetry principle underlying it. These symmetries are partly manifested in the gauge invariance of the theory and also in the duality symmetries such as T-duality and S-duality. A formulation that manifests some of these symmetries would be invaluable. The BRST formulation of string field theory solves the technical problem of constructing a gauge invariant action [1]–[3]. Nevertheless a space-time interpretation is obscured by the heavy reliance on world sheet BRST properties. The sigma model approach [4]-[11]is somewhat more geometrical, but the issue of understanding gauge invariance or of going off-shell is rife with difficulty. Some progress on this has been recently achieved [12, 13]. The loop variable approach motivated by the sigma model was used to obtain the free equations of motion in a simple manner [14]. It had the added advantage that the gauge transformations had a simple space-time interpretation of being scale transformations. Speculations on the symmetry principle of string theory are also contained in [14]. More recently, in papers [15, 16] (hereafter I and II respectively) we had discussed a loop variable approach to gauge invariant string interactions. In this paper we describe some concrete applications of this method.

The basic idea of the loop variable approach is to describe the space time fields in string theory, using a generalized Fourier transform, in terms of a "wave functional" $\Psi[k(s)]$. On mode expanding, the wave functional becomes a function $\Psi[k_1, k_2, \ldots, k_n, \ldots]$, where n runs over all natural numbers. Thus

$$\Phi[X(z+s)] = \int \mathcal{D}k(s) e^{i \oint_c ds X(z+s) k(s)} \Psi[k(s)] \,. \tag{1.1}$$

The free equations of motion are obtained by working with the the loop variable $e^{i \oint_c ds X(z+s) k(s)}$. Actually the modified loop variable $e^{ia \int_c k(s) \partial_z X(z+as) ds + ik_0 X(z)}$ was used earlier and is more appropriate for our purposes [14]. The gauge transformation is given by [14]

$$k(s) \to k(s) \lambda(s) \,. \tag{1.2}$$

The interacting equation was obtained by broadening the loop to a band whose width is parametrized by t, so that k_1's at different t correspond to different strings and at the same t would correspond to the same string. The loop variable thus takes the form:

$$e^{i \int dt \int_c k(s,t) \partial_z X(z+s,t) ds + i \int dt k_0(t) X(t)}$$

and $\Psi[k(s,t)]$ is the form of the wave functional. Now the fully interacting gauge transformation is [15, 16]

$$k(t,s) \to \int dt' \lambda(t', s) k(t,s) \,. \tag{1.3}$$

An important feature is that this symmetry does not depend on world sheet properties and holds even when there is a short distance cutoff on the world sheet.

On mode expanding the wave functional qq becomes $\Psi[k_1(t), k_2(t), \ldots, k_n(t), \ldots]$. This is described in detail in I and II. The wave functional Ψ was assumed to have certain properties defined below, but beyond that, was not discussed at all. In Sec 2 of this paper we discuss Ψ. We give an explicit expression for Ψ. A wave functional with these properties can describe the perturbative equations of motion. In section 3 and section 4 we work out the equation for the first massive mode of the string interacting with electromagnetism. This equation is valid off the free mass shell also. The full equation has an infinite number of terms involving all the massive modes. We keep only the massless and first massive modes. We also show the gauge invariance to leading order in momenta and explain the role of the higher modes in preserving gauge invariance in higher orders.

However if one wants to describe non perturbative equations with some specific functional form for the fields, other forms of Ψ are required. An example is the Born-Infeld equation, describing uniform field strengths [17, 18]. In section 5 we give a construction of Ψ for this situation as well. In this approach, z, the position of the vertex operator, is also labelled by t, i.e. $z(t)$. Since z is real for an open string, it is possible to let $z(t) = t$. This is done in this paper and one makes more direct contact with the open string sigma-model or boundary string field theory approaches [18, 17, 20], [27]–[29].

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2. Ψ for Perturbation theory

Let us recollect the basic properties required of Ψ.

$$\int \left[\prod_{n>0} \mathcal{D}k_n(t)]\Psi[k_0(t), k_n(t)\right] = \prod_t \delta(k_0(t)) \,.$$

In position space this equation would read:

$$\int \left[\prod_{n>0} \mathcal{D}k_n(t)\right] \Psi[X(t), k_n(t)] = \int \left[\prod_{n\geq 0} \mathcal{D}k_n(t)\right] e^{i\int dt k_0(t) X(t)} \Psi[k_0(t), k_n(t)] = 1$$

$$\int \left[\prod_{n>0} \mathcal{D}k_n(t)\right] k_1^\mu(t_1) \Psi[k_0(t), k_n(t)] = A^\mu(k_0(t_1)) \prod_{t\neq t_1} \delta(k_0(t)) \,.$$

In position space:

$$\int \left[\prod_{n>0} \mathcal{D}k_n(t)\right] k_1^\mu(t_1) \Psi[X(t), k_n(t)] = A^\mu(X(t_1)), \tag{2.1}$$

$$\int \left[\prod_{n>0} \mathcal{D}k_n(t)\right] k_1^\mu(t_1) k_1^\nu(t_2) \Psi[k_0(t), k_n(t)] = A^\mu(k_0(t_1)) A^\nu(k_0(t_2)) \prod_{t\neq t_1, t_2} \delta(k_0(t)) + \\ + \delta(t_1 - t_2) S_{1,1}^{\mu\nu}(k_0(t_1)) \prod_{t\neq t_1} \delta(k_0(t)).$$

In position space:

$$\int \left[\prod_{n>0} \mathcal{D}k_n(t)\right] k_1^\mu(t_1) k_1^\nu(t_2) \Psi[X(t), k_n(t)] = A^\mu(X(t_1)) A^\nu(X(t_2)) + \\ + \delta(t_1 - t_2) S_{1,1}^{\mu\nu}(X(t_1)) \,. \tag{2.2}$$

A word on the notation used in these equations: $k_0(t)$ refers to the momentum of the vertex operator located at the point $z(t)(= t$ in this paper). t is only a label. Thus while $X(t)$ is a non-trivial function of t, $k_0(t)$ is not. Further in the end all momenta are integrated over. Thus $k_0(t)$ is only an integration variable: $\int dk_0(t) \phi(k_0(t)) e^{ik_0(t)X} = \phi(X)$. It also follows that $\phi(k_0(t))$ is not explicitly or implicitly a function of t.

We can easily construct a wave functional, Ψ, with these properties. Let us first set $S_{1,1}^{\mu\nu} = 0$. Then the solution is very simple:

$$\Psi[X(t), k_n(t)] = \prod_{t,\mu} \delta[k_1^\mu(t) - A^\mu(X(t))] \prod_{t,n>0,\mu} \delta[k_n^\mu(t)] \,. \tag{2.3}$$

In momentum space one can write

$$\Psi[k_0(t), k_n(t)] = \int \mathcal{D}X(t) e^{-i\int dt k_0(t) X(t)} \prod_{t,\mu} \delta[k_1^\mu(t) - A^\mu(X(t))] \prod_{t,n>0,\mu} \delta[k_n^\mu(t)] \,. \tag{2.4}$$

The generalization, to include the massive modes, $S_n^\mu, n > 1$ is obvious.

One can easily check that Ψ satisfies (2.2). To make sense of the product over t, one has to regularise. This is done below.

Let us introduce $S_{1,1}^{\mu\nu}$. This can be done by broadening the delta function to include a non-trivial two-point function. Let us first discretise the parameter t and set $t = na$ where $n \in \mathbb{Z}$ and a is a regulator. Then the wave functional

$$\Psi[k_1(t), k_m(t), X(t)] = \lim_{N\to\infty} \left[\prod_{n=-N/2}^{N/2} \frac{\prod_{m>0} \delta[k_m(n)]}{\sqrt{2\pi S_{1,1}(X(n))/a}} \right] e^{-\sum_{n=-N/2}^{N/2} \frac{[k_1(n)-A_1(X(n))]^2 a}{2S_{1,1}(X(n))}} . \tag{2.5}$$

This satisfies

$$\begin{aligned} \langle k_1(n_1)\rangle &= A(X(n_1)) \\ \langle k_1(n_1)k_1(n_2)\rangle - \langle k_1(n_1)\rangle\langle k_1(n_2)\rangle &= S_{1,1}(X(n_1))\frac{\delta_{n_1,n_2}}{a} , \end{aligned} \tag{2.6}$$

where $\langle \ldots \rangle = \int [\prod_{n>0} \mathcal{D}k_n] \ldots \Psi[k_n(t), X(t)]$.

We can Fourier transform $X(n)$ and write down the momentum space wave functional

$$\begin{aligned} &\Psi[k_1(t), k_m(t), k_0(t)] = \\ &= \lim_{N\to\infty} \left[\prod_{n=-N/2}^{N/2} dX(n) e^{-iak_0(n)X(n)} \frac{\prod_{m>0} \delta[k_m(n)]}{\sqrt{2\pi S_{1,1}(X(n))/a}} \right] e^{-\sum_{n=-N/2}^{N/2} \frac{[k_1(n)-A_1(X(n))]^2 a}{2S_{1,1}(X(n))}} . \end{aligned} \tag{2.7}$$

From (2.7) it is clear that the momentum variable conjugate to $X(n)$ is $ak_0(n)$ and thus in the inverse Fourier transform, the measure $\mathcal{D}k_0(t)$ is to be interpreted as $\prod_n (a\, dk_0(n))$. Let us call $ak_0 = \bar{k_0}$.

If we define in the usual way

$$\int dX e^{-ikX} A(X) = A(k) .$$

Then

$$\int dX e^{-iakX} A(X) = A(ak) = A(\bar{k_0}) .$$

The wave functional (2.7) satisfies

$$\langle k_1^\mu(n_1) k_0^\nu(n_2)\rangle = A^\mu(\bar{k_0}(n_1))\bar{k_0}^{\nu}(n_1)\frac{\delta_{n_1,n_2}}{a} \prod_{n\neq n_1} \delta[\bar{k_0}(n)] .$$

Here $\bar{k_0}$ is the (physical) momentum conjugate to the position. Since the measure of integration is also $d\bar{k_0}$ this overall factor of a is not important.

Note also that in momentum space,

$$\langle 1 \rangle = \prod_n \delta[\bar{k_0}(n)] .$$

This is the regularized version of the first equation in (2.1), which is the statement that the vacuum is translationally invariant.[1]

We can also include the gauge transformations of the loop variable by incorporating $\lambda_1(t)$, $\lambda_2(t)\ldots$into the wave functional. Let us consider (2.3).

$$\Psi[X(t),k_n(t)] = \prod_{t,\mu}\delta[k_1^\mu(t) - A^\mu(X(t))]\prod_{t,n>0,\mu}\delta[k_n^\mu(t)]\,.$$

One can incorporate $\lambda_1(t)$ by adding a delta function

$$\Psi[X(t),k_n(t),\lambda_1(t)] = \prod_{t,\mu}\delta[k_1^\mu(t) - A^\mu(X(t))]\prod_{t}\delta[\lambda_1(t) - \Lambda(X(t))]\,. \tag{2.8}$$

In momentum space

$$\Psi[k_0(t),k_n(t),\lambda_1(t)] = \int \mathcal{D}X(t)e^{-i\int dt k_0(t)X(t)}\Psi[X(t),k_n(t),\lambda_1(t)]\,.$$

Regularization is understood and has not been explicitly shown.

The above wave functional satisfies

$$\begin{aligned}\langle\lambda_1(n_1)\rangle &= \Lambda(\bar{k_0}(n_1))\\ \langle\lambda_1(n_1)k_0^\mu(n_2)\rangle &= \Lambda(\bar{k_0}(n_1))\bar{k_0}^\mu(n_2)\frac{\delta_{n_1,n_2}}{a}\prod_{n\neq n_1}\delta[\bar{k_0}(n)]\end{aligned} \tag{2.9}$$

as required. λ_n , $n>2$, can also similarly be included.

3. Gauge transformations of space-time fields

In this section we show that gauge transformations can be consistently defined for space-time fields.

In II, a scheme was described for defining gauge transformations on space-time fields. The basic idea was to first move all vertex operators to one location and then use these vertex operators to define combinations of fields (and loop variables) whose gauge transformations are then worked out.

Thus

$$e^{i\sum_n k_n(z_1)\tilde{Y}_n(z_1)} = e^{i\sum_n K_n(z_1,z-z_1)\tilde{Y}_n(z)}\,.$$

This equation defines the K_n. The precise expressions for $K_n(z_1,z-z_1)$ are given in II. All the z_i dependence is thus made explicit. Then we consider combinations of the form

$$\prod_{i=1}^{N}K_{n_i}(z_i,z-z_i)\tilde{Y}_{n_i}(z)\,.$$

[1] We have set the tachyon field to zero in this paper.

This is used to define the transformation laws for the highest level field $\langle k_{n_1}k_{n_2}\ldots \ldots k_{n_N}\rangle = S_{n_1,n_2,\ldots,n_N}$ in terms of the gauge transformation of lower fields. In this way gauge transformation laws for all fields can be recursively defined.

The crucial point in this construction is that we are using a regularized greens function that has no short distance singularity. This is what allows us to move all vertex operators to one point, and also allows us to contract them if necessary. Furthermore we will also integrate the location of the vertex operator over some (finite) range - this is the usual Koba-Nielsen integration. In the examples considered in [16] the z-dependence somehow dropped out and it wasn't necessary to do the integrals. In general, the z-dependence will not drop out (indeed, they are essential) and only the integrated equation makes sense. The combination of operators used here for defining the gauge transformations are precisely the combinations that occur in any equation of motion - along with contractions of the $\tilde{Y}$'s. Since this contraction does not affect the consistency of this procedure it follows that there is a well defined map from loop variables and their gauge transformation to space-time fields and their gauge transformation. Thus gauge invariance at the level of loop variables implies gauge invariance of the field equations.

Below we discuss the gauge invariance of the Y_2 equation of [15]. We will demonstrate the leading order gauge invariance explicitly. For the non-leading term we will show how massive modes contribute in a crucial way by providing the types of terms required for gauge invariance. Explicit calculations of numerical coefficients is rather tedious and will not be attempted.

We will include the contributions of some of the massive modes and will go some way in proving the gauge invariance of that equation - far enough to make the pattern clear.

We use $ln\,(\epsilon^2+(z_1-z_2)^2)$ as the two point function $\langle X(z_1)X(z_2)\rangle$. If we use this, the equation of motion corresponding to Y_2 from [15] (combining equations (5.3.39),(5.3.40), (5.3.45), (5.3.50) , (5.3.51) and (5.3.54)) becomes the sum of various terms listed below:

$$-A(p)\cdot(p+q)iA^\mu(q)|\frac{(z-w)^2}{\epsilon^2}+1|^{p\cdot q}(\epsilon)^{(p+q)^2}Y_2^\mu e^{i(p+q)Y} \tag{3.1}$$

$$-S_{1,1}^{\mu\nu}k_0^\nu iY_2^\mu e^{ik_0Y}(\epsilon)^{k_0^2} \tag{3.2}$$

$$-(\epsilon)^{k_0^2}S_2(k_0)\cdot k_0 ik_0^\mu Y_2^\mu e^{ik_0Y} \tag{3.3}$$

$$(\epsilon)^{(p+q)^2}|\frac{(z-w)^2}{\epsilon^2}+1|^{p\cdot q}A(p)\cdot A(q)i(p+q)^\mu Y_2^\mu e^{ik_0Y} \tag{3.4}$$

$$(\epsilon)^{k_0^2}S_{1,1\nu}^{\nu}(k_0)ik_0^\mu Y_2^\mu e^{ik_0Y} \tag{3.5}$$

$$(\epsilon)^{k_0^2}k_0^2 iS_2^\mu(k_0)Y_2^\mu e^{ik_0Y}\ . \tag{3.6}$$

The notation is that of I and II. The reader need only note that after obtaining the equations of motion one can set $x_n=0$ (again in the notation of I and II) and after that $Y_2\equiv\tilde{Y}_2\equiv\partial_z^2X$ and $Y\equiv X$.

The gauge transformations of the fields are (according to the method defined above and in [16]) :

$$\begin{aligned}\delta S^{\mu\nu}_{1,1}(k)e^{ik_0Y} &= [\Lambda^{(\mu}_{1,1}(k)k^{\nu)}_0 + \delta_{int}S^{\mu\nu}_{1,1}(k)]e^{ik_0Y} \\ \delta A^{\mu}_1(p)e^{ip_0Y} &= p^\mu \Lambda_1(p)e^{ip_0Y}\end{aligned} \tag{3.7}$$

and

$$\delta_{int}S^{\mu\nu}_{1,1}(k) = \int dpdq\delta(p+q-k)[\Lambda_1(p)q^{(\nu}A^{\mu)}_1(q)] \tag{3.8}$$

$$\delta S^{\mu}_2(k_0) = \Lambda_2(k_0)k^\mu_0 + \Lambda^\mu_{1,1}(k_0) + \delta_{int}S^\mu_2(k_0) \tag{3.9}$$

with

$$\delta_{int}S^{\mu}_2(k_0) = \int dpdq\delta(p+q-k_0)\Lambda_1(p)A^\nu_1(q)\,. \tag{3.10}$$

Let us write down the contribution of the various terms:

$$\begin{aligned}
\delta\ (3.1) &= (-p\cdot(p+q)\Lambda(p)iA^\mu(q) - A(p)\cdot(p+q)iq^\mu\Lambda(q))\left[1+\frac{(z-w)^2}{\epsilon^2}\right]^{p\cdot q}\epsilon^{(p+q)^2} \\
\delta\ (3.2) &= -\Lambda^{(\mu}_{1,1}k^{\nu)}_0 k^\nu_0 i\epsilon^{k_0^2} - \Lambda_1(p)q^{(\nu}A^{\mu)}_1(q)(p+q)^\nu i\epsilon^{(p+q)^2} \\
\delta\ (3.3) &= -\Lambda_2(k_0)k_0^2ik^\mu_0\epsilon^{k_0^2} - \Lambda_{1,1}(k_0)\cdot k_0ik^\mu_0\epsilon^{k_0^2} - \Lambda_1(p)A_1(q)\cdot(p+q)i(p+q)^\mu\epsilon^{(p+q)^2} \\
\delta\ (3.4) &= 2p\cdot A(q)\Lambda(p)i(p+q)^\mu\left[1+\frac{(z-w)^2}{\epsilon^2}\right]^{p\cdot q}\epsilon^{(p+q)^2} \\
\delta\ (3.5) &= 2\Lambda_{1,1}(k_0)\cdot k_0ik^\mu_0\epsilon^{k_0^2} + 2\Lambda_1(p)q\cdot A_1(q)i(p+q)^\mu\epsilon^{(p+q)^2} \\
\delta\ (3.6) &= k_0^2i\Lambda_2(k_0)k^\mu_0\epsilon^{k_0^2} + i\Lambda^\mu_{1,1}(k_0)k_0^2\epsilon^{k_0^2} + i\Lambda(p)A^\mu_1(q)(p+q)^2\epsilon^{(p+q)^2}\,.
\end{aligned}$$

In the above expressions, integration over momenta p, q with momentum conserving delta function, and the Koba-Nielsen variables w, z with suitable regularization, are understood.

Terms involving Λ_2 and $\Lambda_{1,1}$ can easily be seen to add up to zero. We are left with

$$\begin{aligned}
&A(p)\cdot(p+q)\Lambda_1(q)\left[-iq^\mu\left[1+\frac{(z-w)^2}{\epsilon^2}\right]^{p\cdot q} - ip^\mu - i(p+q)^\mu\right] + \\
&\quad +A(p)\cdot p\Lambda_1(q)2i(p+q)^\mu + A(p)\cdot q\Lambda_1(q)2i(p+q)^\mu\left[1+\frac{(z-w)^2}{\epsilon^2}\right]^{p\cdot q}
\end{aligned} \tag{3.11}$$

and

$$iA^\mu(q)\Lambda_1(p)\left[-p\cdot(p+q)\left[1+\frac{(z-w)^2}{\epsilon^2}\right]^{p\cdot q} - q\cdot(p+q) + (p+q)^2\right]. \tag{3.12}$$

If one expands $[1+(z-w)^2/\epsilon^2]^{p\cdot q} \approx 1 + p\cdot q(z-w)^2/\epsilon^2 + \ldots O((z-w)^4)$ in the above expressions, it is easy to see that the leading terms cancel leaving uncancelled terms of order $(z-w)^2$.

Where does one find contributions to cancel these terms? The answer is that they come from the variations of higher modes such as $S^{\mu\nu\rho}_{1,1,2}$. These higher modes do contribute to the Y_2 equation through the following terms in the loop variable

$$\int\int\int\int dt_1 dt_2 dt_3 dt_4 k_1(t_1)\cdot k_1(t_2)\frac{\partial^2[\tilde{\Sigma}+\tilde{G}](t_1,t_2)}{\partial x_1(t_1)\partial x_1(t_2)}$$
$$k_2(t_3)\cdot k_0(t_4)\frac{\partial[\tilde{\Sigma}+\tilde{G}](t_3,t_4)}{\partial x_2(t_3)}e^{\int dt\int dt' k_0(t)\cdot k_0(t')[\tilde{\Sigma}+\tilde{G}](t,t')}e^{i\int k_0 Y} \tag{3.13}$$

and also

$$\int\int dt_1 dt_2 k_1(t_1)\cdot k_1(t_2)\frac{\partial^2[\tilde{\Sigma}+\tilde{G}](t_1,t_2)}{\partial x_1(t_1)\partial x_1(t_2)}$$
$$\int dt_3 e^{\int dt\int dt' k_0(t)\cdot k_0(t')[\tilde{\Sigma}+\tilde{G}](t,t')} ik_2^{\nu}(t_3)Y_2^{\mu}(t_3)e^{i\int k_0 Y} \tag{3.14}$$

using

$$\begin{aligned}\frac{\partial^2\tilde{\Sigma}(t_1,t_2)}{\partial x_1(t_1)\partial x_1(t_2)}\ |_{x_n(t_1)=x_n(t_2)} &\approx \frac{1}{2}(\frac{\partial^2}{\partial x_1^2}-\frac{\partial}{\partial x_2})\Sigma\,,\\ \frac{\partial}{\partial x_2(\sigma_1)}\tilde{\Sigma}\ |_{x_n(t_1)=x_n(t_2)} &\approx \frac{1}{2}\frac{\partial}{\partial x_2}\Sigma\,,\\ \frac{\partial}{\partial x_2}\tilde{G}\ |_{t_1=t_2,x_n=0} &= \frac{1}{\epsilon^2}\,,\end{aligned} \tag{3.15}$$

and

$$\frac{\partial^2\tilde{G}(t_1,t_2)}{\partial x_1(t_1)\partial x_1(t_2)}\ |_{x_n=0)}= -\frac{1}{\epsilon^2}$$

we get on varying Σ in (3.13) and taking expectation of the loop variable,

$$\frac{3}{2\epsilon^2}\langle k_1\cdot k_1 k_2\cdot k_0 ik_0^{\mu}\rangle Y_2^{\mu}\approx\frac{3}{2\epsilon^2}S^{\rho\rho\nu}_{1,1,2}(k_0)k_0^{\nu}ik_0^{\mu}Y_2^{\mu}\,, \tag{3.16}$$

where we have kept only the S_{112} "contact" term in the expectation value.

The second term, (3.14), gives analogously

$$-\frac{1}{\epsilon^2}\langle k_1\cdot k_1 k_0^2 ik_2^{\nu}\rangle Y_2^{\mu}\approx -\frac{1}{\epsilon^2}k_0^2 S^{\rho\rho\mu}_{1,1,2}(k_0)Y_2^{\mu}\,. \tag{3.17}$$

There are also terms of the form $k_1\cdot k_2 k_1\cdot k_0$, $k_1\cdot k_2 k_1^{\mu}$ that can contribute to the Y_2 equation. Since we are only interested in showing that these terms contribute $O(z^2)$ terms to the equation of motion, and not in the precise coefficient, we will not worry about it here.

Let us now turn to the scheme of [16] to evaluate $\delta S^{\mu\nu\rho}_{1,1,2}$.

Let us start with $K_1^\mu(z_1, z-z_1)K_1^\nu(z_2, z-z_2)\tilde{Y}_1^\mu(z)\tilde{Y}_1^\nu(z)$.

$$
\begin{aligned}
&\int dz_1 \int dz_2 \langle K_1^\mu(z_1, z-z_1) K_1^\nu(z_2, z-z_2)\rangle = \\
&\quad = \int dz_1 \int dz_2 \langle (k_1^\mu(z_1) + (z_1 - z)k_0^\mu)(k_1^\nu(z_2) + (z_2 - z)k_0^\nu)\rangle \\
&\quad = \int dz_1 \int dz_2 S_{1,1}^{\mu\nu}(k_0(z_1))\delta(z_1 - z_2) + A_1^\mu(k_0(z_1))A_1^\nu(k_0(z_2)) \\
&\qquad + (z_1 - z)k_0^\mu(z_1)A_1^\nu(k_0(z_2))\delta(z_1 - z_2) + (z_2 - z)k_0^\nu(z_2)A_1^\mu(k_0(z_1))\delta(z_1 - z_2) \\
&\quad = \int dz_1 S_{1,1}^{\mu\nu}(k_0(z_1)) + \int dz_1 \int dz_2 A_1^\mu(k_0(z_1))A_1^\nu(k_0(z_2)) \\
&\qquad + \int dz_1 (z_1 - z) k_0^{(\mu}(z_1) A_1^{\nu)}(k_0(z_1))\,.
\end{aligned} \tag{3.18}
$$

We remind the reader about the following two points. First, $k_0(z)$ refers to a momentum of a field associated with a particular vertex operator located at z. k_0 is *not* a function of z. In the same way $S_{1,1}(k_0(z))$ is a field with momentum $k_0(z)$ — it is not a function, implicitly or explicitly, of z. The z-dependence in this approach is all explicitly there in the equations because we have extracted all the z-dependences by Taylor expanding the vertex operators. Second, integration of z_i over some suitable range is understood in all these equations.[2]

We now consider the variation of both sides.

$$
\begin{aligned}
&\delta \int dz_1 \int dz_2 \langle (k_1^\mu(z_1) + (z_1 - z)k_0^\mu)(k_1^\nu(z_2) + (z_2 - z)k_0^\nu)\rangle = \\
&\qquad = \int dz' \int dz_1 \int dz_2 \langle \lambda_1(z')(k_0^\mu(z_1)(k_1^\nu(z_2) + (z_2 - z)k_0^\nu) + (k_1^\mu(z_1) + \\
&\qquad\qquad + (z_1 - z)k_0^\mu)k_0^\nu(z_2))\rangle\,.
\end{aligned} \tag{3.19}
$$

Thus variation of the space-time fields gives:

$$
\begin{aligned}
&\delta\Big\{\int dz_1 S_{1,1}^{\mu\nu}(k_0(z_1)) + \int dz_1 \int dz_2 A_1^\mu(k_0(z_1))A_1^\nu(k_0(z_2)) + \\
&\quad + \int dz_1 (z_1 - z)k_0^{(\mu}(z_1)A_1^{\nu)}(k_0(z_1))\Big\} = \\
&\qquad = \int dz_1 [\delta S_{1,1}^{\mu\nu}(k_0(z_1))] + \int dz_1 \int dz_2 [\Lambda_1(k_0(z_1))k_0^{(\mu}(z_1)A_1^{\nu)}(k_0(z_2))] + \\
&\qquad\quad + \int dz_1 [(z_1 - z)k_0^{(\mu}(z_1)\Lambda_1(k_0(z_1))k_0^{\nu)}(z_1)]\,.
\end{aligned} \tag{3.20}
$$

---

[2] Thus $\int dz_1 \equiv \int_a^R dz_1$ for some a, R that we will leave unspecified. If we are working on a unit disk then z_1 has a range of 2π. If on the upper half plane, z_1 ranges from $-\infty$ to $+\infty$. In the context of the proper time formalism, z would range from 0 to R. This scheme dependence is expected in any off shell description.

Whereas variation of the loop variable momenta yields

$$\int dz'\Lambda_{1,1}^{(\nu}(k_0(z'))k_0^{\mu)}(z') + \int dz'dz''\Lambda_1(k_0(z'))A_1^{(\nu}(k_0(z'))[k_0(z') + k_0(z'')]^{\mu)} + \\ + \int dz' 2\Lambda_1(k_0(z'))k_0^{\mu}(z')k_0^{\nu}(z')(z'-z)\,. \tag{3.21}$$

Comparison of the two equations gives:

$$\int dz_1 \delta S_{1,1}^{\mu\nu}(k_0(z_1)) = \int dz'\Lambda_{1,1}^{(\nu}(k_0(z'))k_0^{\mu)}(z') + \\ + \int dz_1 dz_2 \Lambda_1(k_0(z_1))A_1^{(\nu}(k_0(z_2))k_0^{\mu)}(z_2)\,. \tag{3.22}$$

One can extract from this $\delta S_{1,1}^{\mu\nu}(p)$ if one uses the fact that the integral over z' in the LHS of (3.22) gives simply $range\,of\;z' \times \delta S_{1,1}^{\mu\nu}$.

As pointed out in [16] earlier, there are no z-dependent terms in this variation. However when we turn to $S_{1,1,2}$ this will not be true.

Let us turn to $S_{1,1,2}$.

We start with

$$\int dz_1 \int dz_2 \int dz_3 \langle K_1^{\mu}(z_1, z-z_1)K_1^{\nu}(z_2, z-z_2)K_2^{\rho}(z_3, z-z_3)\rangle = \\ = \Big\langle (k_1^{\mu}(z_1) + (z_1 - z)k_0^{\mu}(z_1))(k_1^{\nu}(z_2) + (z_2 - z)k_0^{\nu}(z_2))(k_2^{\rho}(z_3) + (z_3 - z)k_1^{\rho} + \\ + \frac{(z_3-z)^2}{2}k_0^{\rho})\Big\rangle\,. \tag{3.23}$$

We will focus our attention on the $O(z^2)$ terms, which are:

$$\Big\langle k_1^{\mu}(z_1)k_1^{\nu}(z_2)k_0^{\rho}(z_3)\frac{(z_3-z)^2}{2} + k_0^{\mu}(z_1)k_0^{\nu}(z_2)k_2^{\rho}(z_3)(z_1-z)(z_2-z) + \\ + k_0^{\mu}(z_1)k_1^{\nu}(z_2)k_1^{\rho}(z_3)(z_1-z)(z_3-z) + k_1^{\mu}(z_1)k_0^{\nu}(z_2)k_1^{\rho}(z_3)(z_3-z)(z_2-z)\Big\rangle\,. \tag{3.24}$$

Taking the expectation value one gets (Integration over z_i is understood):

$$S_{1,1}^{\mu\nu}(k_0(z_1))k_0^{\rho}(z_1)\frac{(z_1-z)^2}{2} + \\ + A_1^{\mu}(k_0(z_1))A_1^{\nu}(k_0(z_2))\left[k_0^{\rho}(z_1)\frac{(z_1-z)^2}{2} + k_0^{\rho}(z_2)\frac{(z_2-z)^2}{2}\right] + \\ + \{S_{1,1}^{\nu\rho}(k_0(z_1))k_0^{\mu}(z_1)(z_1-z)^2 + \\ + A_1^{\nu}(k_0(z_2))A_1^{\rho}(k_0(z_3))[k_0^{\mu}(z_2)(z_2-z) + k_0^{\mu}(z_3)(z_3-z)](z_3-z) + \\ + \mu \leftrightarrow \nu\} + \\ + S_2^{\rho}(z_2)k_0^{\mu}(z_3)k_0^{\nu}(z_3)(z_3-z)^2\,. \tag{3.25}$$

For clarity we will rewrite this expression with momenta $p \cdot q$. This gives

$$
\begin{aligned}
& S^{\mu\nu}_{1,1}(p) p^\rho \frac{(z_1 - z)^2}{2} + \\
& \quad + A^\mu_1(p) A^\nu_1(q) \left[p^\rho \frac{(z_1 - z)^2}{2} + q^\rho \frac{(z_2 - z)^2}{2} \right] + \\
& \quad + \{ S^{\nu\rho}_{1,1}(p) p^\mu (z_1 - z)^2 + A^\nu_1(q) A^\rho_1(p) [q^\mu (z_2 - z) + p^\mu (z_3 - z)](z_3 - z) + \\
& \qquad + \mu \leftrightarrow \nu \} + \\
& \quad + S^\rho_2(p) p^\mu p^\nu (z_3 - z)^2 \,.
\end{aligned} \tag{3.26}
$$

We have to compare the gauge transformation of this expression (using the previously determined variations of the fields), and compare with what one gets on varying the loop variable expression directly. If they are not equal, the difference will be assigned to the gauge transformation of $S_{1,1,2}$. Let us focus on terms having the structure $A^\nu_1(q)\Lambda_1(p)p^\mu q^\rho$.

Note that in (3.26) there are two contributions that have this structure: One is multiplied by $(z_3 - z)^2$ and the other by $(z_2 - z)^2$, which is the same since z_i are integration variables.

Now we vary the k_i in (3.24) and look at the terms that can give A^ν_i. These are:

$$
\int dz' \left\langle \lambda_1(z') (k^\mu_0(z_1) k^\nu_1(z_2) k^\rho_0(z_3) \frac{(z_3 - z)^2}{2} + k^\mu_0(z_1) k^\nu_1(z_2) k^\rho_0(z_3)(z_1 - z)(z_3 - z)) \right\rangle . \tag{3.27}
$$

The second term gives a contribution proportional to $\Lambda_1(p)p^\mu A^\nu_1(q) q^\rho (z_1 - z)(z_2 - z)$. which is clearly different from anything obtained from (3.26). This term has thus to be assigned to $\delta S^{\mu\nu\rho}_{1,1,2}$. Thus we have shown that there are $O(z^2)$ terms in the gauge variation of $S_{1,1,2}$ contribution to the Y_2 equation of motion.

The above calculation gives us a picture of how the contributions from the higher modes to a given equation of motion add up to reproduce expressions of the form $\int dz \int dw (z-w)^{2p\cdot q}$. As mentioned in [16] , the fact that the equations are gauge invariant follows from the gauge invariance of the loop variable expression and the fact that the above map — from loop variables to space-time fields and their respective gauge transformations — is well defined. Furthermore the space-time gauge invariance does not depend on any world sheet properties such as BRST invariance, so this is possible. This also allows one to go off-shell without violating gauge invariance.

4. Dimensional reduction

We can dimensionally reduce the above equations to obtain equations for the massive modes with masses equal to that in string theory. This is where we make contact with string theory. The prescription was given in [16]. We start with the loop variable:

$$\exp\Big\{\int d\sigma_1 \int d\sigma_2 \sum_{n,m\geq 0} (k_n(\sigma_1)\cdot k_m(\sigma_2)\frac{\partial^2[\tilde{G}+\tilde{\Sigma}](\sigma_1,\sigma_2)}{\partial x_n(\sigma_1)\partial x_m(\sigma_2)} + k_{n,V}(\sigma_1)k_{m,V}(\sigma_2)\times$$
$$\times\frac{\partial^2\tilde{\Sigma}(\sigma_1,\sigma_2)}{\partial x_n(\sigma_1)\partial x_m(\sigma_2)})\Big\} : \exp\Big\{i\int d\sigma \sum_{n\geq 0} k_n Y_n(z(\sigma)\Big\} : . \tag{4.1}$$

The extra dimension (27th in the bosonic string, or 11th in the superstring) is denoted by V in the above equation. In [16] we set $k_0^V = \sqrt{P-1}/N$ where N is the number of fields in a particular term in the equation of motion. (Thus $N=1$ for the free theory, $N=2$ in the quadratic term of an equation of motion, etc.) and P is the engineering dimension of the vertex operator for which the equation is being written. Thus $P=0$ for e^{ikX}, $P=1$ for $\partial_z X e^{ikX}$, etc. Note that if some of the Y's are contracted then we get powers of ϵ, and P must count these powers also as contributing to the engineering dimension. Furthermore we will let

$$\langle k_n^V\rangle = S_n^V$$

The factor $1/N$ in k_0^V can also be thought of as being equivalent to the imposition of momentum conservation in the extra dimension so that the total momentum of each term in an equation adds up to the same value as for the linear term, where it is set equal to the mass of the field.

Let us apply this to the first massive level equation of the last section. This has $P=2$.

First we have to establish the gauge transformation laws for the fields A_1^V, $S_{1,1}^{VV}$, $S_{1,1}^{\mu V}$, and S_2^V. We start with

$$\begin{aligned}\langle K_1^V(z_1,z-z_1)\rangle Y_1^V &= \langle (k_1^V(z_1)+(z_1-z)k_0^V)\rangle Y_1^V \\ &= A_1^V(k_0(z_1))Y_1^V \end{aligned}\tag{4.2}$$

$$\delta\langle K_1^V(z_1,z-z_1)\rangle Y_1^V = \int dz'\langle\lambda_1(z')k_0^V(z_1)\rangle Y_1^V\,. \tag{4.3}$$

Since $P=1$, $k_0^V=0$ in the above equation. So,

$$\delta A_1^V(k_0) = 0\,. \tag{4.4}$$

This is just as well, since for making contact with string theory, we will set this field to zero. Note also, that if Y_1^V in (4.3) is contracted with Y^V in $e^{ik_0^V Y^V}$ we get (Using $\langle Y_1^V(z)Y^V(z)\rangle = \epsilon_1 \approx 1/\epsilon$)

$$\delta\Big\langle\int dz K_1^V(z_1,z-z_1)k_0^V(z)\Big\rangle\epsilon_1 = \int dz'\Big\langle\int dz\lambda_1(z')k_0^V(z_1)k_0^V(z)\Big\rangle\epsilon_1 = 0$$

since $(P-1)\epsilon_1 = 0$. Thus the above procedure is consisitent with contractions.

We now turn to

$$\int dz_1 \int dz_2 \langle K_1^V(z_1, z-z_1) K_1^V(z_2, z-z_2)\rangle Y_1^V Y_1^V = \tag{4.5}$$
$$= \int dz_1 \int dz_2 \langle (k_1^V(z_1) + (z_1 - z)k_0^V)(k_1^V(z_2) + (z_2 - z)k_0^V)\rangle Y_1^V Y_1^V$$
$$= \Big[\int dz_1 \int dz_2 S_{1,1}^{VV}(k_0(z_1))\delta(z_1 - z_2) + A_1^V(k_0(z_1))A_1^V(k_0(z_2)) + (z_1 - z) \times$$
$$\times k_0^V(z_1)A_1^V(k_0(z_2))\delta(z_1 - z_2) + (z_2 - z)k_0^V(z_2)A_1^V(k_0(z_1))\delta(z_1 - z_2)\Big] Y_1^V Y_1^V .$$

Since $A_1^V = 0$ we have

$$= \int dz_1 S_{1,1}^{VV}(k_0(z_1)) Y_1^V Y_1^V . \tag{4.6}$$

We turn to the variations.

$$\delta\langle K_1^V K_1^V\rangle Y_1^V Y_1^V = \int dz' \int dz_1 \int dz_2 \langle \lambda(z')(k_0^V(z_1)k_1^V(z_2) + k_1^V(z_1)k_0^V(z_2) +$$
$$+ 2k_0^V(z_1)k_0^V(z_2)(z_2 - z))\rangle Y_1^V Y_1^V$$
$$= 2\int dz_2 \Lambda_{1,1}^V(z_2) Y_1^V Y_1^V + 2\int dz' \underbrace{k_0^V(z')}_{1}\underbrace{k_0^V(z')}_{1}(z' - z) \times$$
$$\times \Lambda_1(z') Y_1^V Y_1^V$$
$$= 2\int dz_2 [\Lambda_{1,1}^V(z_2) + (z_2 - z)\Lambda_1(z_2)] Y_1^V Y_1^V . \tag{4.7}$$

Again we have set $A_1^V = 0$.
Thus we get

$$\delta \int dz_2 S_{1,1}^{VV} = 2\int dz_2 [\Lambda_{1,1}^V(z_2) + (z_2 - z)\Lambda_1(z_2)] . \tag{4.8}$$

We now turn to

$$\int dz_1 \int dz_2 \langle K_1^V(z_1, z - z_1) K_1^\mu(z_2, z - z_2)\rangle Y_1^V Y_1^\mu =$$
$$= \int dz_1 \int dz_2 \langle (k_1^V(z_1) + (z_2 - z)k_0^V(z_1))(k_1^\mu(z_2) + (z_1 - z)k_0^\mu(z_2))\rangle Y_1^V Y_1^\mu$$
$$= \Big[\int dz_1 S_{1,1}^{\mu V}(k_0(z_1)) + \int dz_2 (z_2 - z) A_1^\mu(k_0(z_2))\Big] Y_1^\mu Y_1^V . \tag{4.9}$$

Consider the variations of both sides.

$$\delta\langle K_1^V K_1^\mu\rangle Y_1^V Y_1^\mu =$$
$$= \int dz' \int dz_1 \int dz_2 \langle \lambda(z')(k_0^V(z_1)k_1^\mu(z_2) + k_1^V(z_1)k_0^\mu(z_2) +$$

$$+ 2k_0^V(z_1)k_0^\mu(z_2)(z_2-z))\rangle Y_1^V Y_1^\mu$$

$$= \int dz'(\Lambda_{1,1}^\mu + \Lambda_{1,1}^V(z')k_0^\mu + 2\Lambda_1(z')(z'-z)k_0^\mu) +$$

$$+ \int dz' \int dz_1 \Lambda_1(z') A_1^\mu(z_1) \underbrace{[k_0^V(z') + k_0^V(z_1)]}_{1} Y_1^V Y_1^\mu \,. \qquad (4.10)$$

We also have

$$\int dz_2 \delta A_1^\mu(k_0)(z_2-z) = \int dz_2 \Lambda_1 k_0^\mu(z_2-z)\,.$$

Thus, comparing as before, the gauge variations of both sides, we find

$$\int dz_2 \delta S_{1,1}^{V\mu}(k_0) = \int dz_2[\Lambda_{1,1}^\mu(k_0) + \Lambda_{1,1}^V(k_0)k_0^\mu + \Lambda_1 A_1^\mu + \Lambda_1 k_0^\mu(z_2-z)]\,. \qquad (4.11)$$

Finally, consider

$$\langle K_2^V(z_1, z-z_1)\rangle Y_2^V = S_2^V Y_2^V\,.$$

Varying we get

$$\delta\langle K_2^V Y_2^V\rangle = \int dz'\langle \lambda_1(z')[k_1^V(z_1) + (z_1-z)k_0^V(z_1)] + \lambda_2(z')k_0^V(z_1)\rangle Y_2^V$$

$$= \int dz_1[\Lambda_{1,1}^V(z_1) + (z_1-z)\Lambda_1(z_1) + \Lambda_2(z_1)]Y_2^V\,. \qquad (4.12)$$

Comparing gauge variations on both sides gives:

$$\int dz_1 \delta S_2^V = \int dz_1[\Lambda_{1,1}^V + \Lambda_2 + (z_1-z)\Lambda_1]\,. \qquad (4.13)$$

At this point one can make the following observation: If we set

$$\int dz'[\Lambda_{1,1}^V + (z'-z)\Lambda_1] = \int dz' \Lambda_2$$

we see that

$$\delta S_{1,1}^{V\mu} = \delta S_2^\mu$$
$$\delta S_{1,1}^{VV} = \delta S_2^V\,. \qquad (4.14)$$

Thus as discussed in [2, 14] for the free theory, we can reduce the number of degrees of freedom in this theory to match that of critical string theory by setting $\langle k_1^V\rangle = A_1^V = 0$, $S_{1,1}^{VV} = S_2^V$ and $S_{1,1}^{V\mu} = S_2^\mu$ consistently.

These identifications follow from the identification at the loop variable level:

$$K_1(z_1, z-z_1)^V \lambda_1(z_2) = \lambda_2(z_1)$$

and

$$k_1^V k_1^\mu = k_2^\nu$$

and

$$k_1^V k_1^V = k_2^V \ .$$

This is very similar to what was done at the free level in [14].

We go back to the loop variable equation for Y_2^μ from [15] ((5.3.39),(5.3.40), (5.3.45), (5.3.50) , (5.3.51) and (5.3.54)) We consider the terms that involve a contraction of the 'V' index. These are

$$\begin{aligned}
&\Big\langle -e^{\int\int k_0(z_3)\cdot k_0(z_4)\tilde{G}}\int dz_1 \int dz_2 k_1(z_1)\cdot k_0(z_2) i \int dz k_1^\mu Y_2^\mu e^{i\int k_0 Y}\Big\rangle = \\
&\qquad = \int [-iA_1^V(p)A_1^\mu(q)(z_1-z_2)^{2p\cdot q} e^{i(p+q)Y}\epsilon^{(p+q)^2} - iS_{1,1}^{V\mu}e^{ikY}\epsilon^{k^2}]Y_2^\mu \\
&\qquad = -iS_{1,1}^{V\mu}\int e^{ikY}\epsilon^{k^2}Y_2^\mu \ , \qquad (4.15)
\end{aligned}$$

where we have set $A_1^V = 0$.

$$\begin{aligned}
&\Big\langle -e^{\int\int k_0(z_3)\cdot k_0(z_4)\tilde{G}}\int dz_1 \int dz_2 k_2(z_1)^v k_0^V(z_2) i k_0^\mu \int Y_2^\mu e^{i\int k_0 Y}\Big\rangle = \\
&\qquad = -S_2^V i k_0^\mu \epsilon^{k^2}\int e^{ikY}Y_2^\mu \qquad (4.16) \\
&\Big\langle -e^{\int\int k_0(z_3)\cdot k_0(z_4)\tilde{G}}\int dz_1 \int dz_2 k_1(z_1)^V k_1^V(z_2) i k_0^\mu \int Y_2^\mu e^{i\int k_0 Y}\Big\rangle = \\
&\qquad = \int [\epsilon^{k^2}S_{1,1}^{VV}(k)\imath k_0^\mu + \epsilon^{(p+q)^2}A_1^V(p)A_1^V(q) i(p+q)^\mu e^{i(p+q)Y}(z-w)^{2p\cdot q}]Y_2^\mu \\
&\qquad = \epsilon^{k^2}S_{1,1}^{VV}(k) i k_0^\mu \int Y_2^\mu e^{ikY} \qquad (4.17) \\
&\Big\langle -e^{\int\int k_0(z_3)\cdot k_0(z_4)\tilde{G}} k_0^V k_0^V i k_2^\nu \int e^{i\int k_0 Y}Y_2^\mu\Big\rangle = \\
&\qquad = \epsilon^{k^2} i S_2^\mu \int e^{ik_0 Y}Y_2^\mu \ . \qquad (4.18)
\end{aligned}$$

When we apply the variations (4.4), (4.8), (4.13), (4.11) to (4.15), (4.16), (4.17) and (4.18), we find that they add up to zero.[3] Thus the full (tree level) equations of string theory for the vertex operator $Y_2^\mu e^{ikY}$ are the sum of the terms in the last section (with indices running 26 dimensions) plus the terms calculated in this section.

This illustrates the dimensional reduction and compatibility with gauge invariance. As explained earlier the contribution of all the infinite number of fields have to be included in order for the cancellation to be exact.

[3]Note also that they add upto zero, (to leading order in $z-w$) even if we don't work with the reduced set of fields, i.e. even without setting $A_1^V = 0$ or making any of the other identifications, the equations are gauge invariant. One can speculate that this describes some version of a non critical string theory.

5. Uniform electromagnetic field: Born-Infeld action

We discuss, in this section, the loop variable approach in a non-perturbative setting. This is the situation of a uniform electromagnetic field. In the sigma model approach this can be done to all orders in the field strength. This gives the Born-Infeld action. More precisely, it gives the β -function, which, as emphasised in [19, 17, 20, 21], is only *proportional* to the equation of motion. The proportionality constant is the Zamolodchikov metric [22]. We will show in this section how this is done in the loop variable approach. As in the perturbative case it involves writing down a suitable wave functional. The main purpose of this section is to reproduce, using loop variables, the results in the literature that have been obtained using the sigma model or more precisely boundary conformal field theory techniques. Gauge invariance is not an issue here since only the field strength enters the equation of motion.

The loop variable approach involves writing down (we set $x_n = 0$ since we are not concerned with gauge invariance) the following object:

$$\int\int \mathcal{D}k_0^\mu(t)\mathcal{D}k_1^\mu(t)e^{-i\int dt[k_0^\mu(t)X^\mu(t)+k_1^\mu(t)\partial_t X^\mu(t)]}$$
$$e^{\frac{1}{2}\int\int dtdt'[\sum_{n,m=0,1}k_m^\mu(t)\mathcal{G}^{\mu\nu}_{m,n}(t,t')k_n^\nu(t')]}\Psi[k_0(t),k_1(t)]\,. \tag{5.1}$$

Here $\mathcal{G}$ is the matrix

$$\begin{pmatrix} G & 0 & \partial_{t'}G & 0 \\ 0 & G & 0 & \partial_{t'}G \\ \partial_t G & 0 & \partial_t\partial_{t'}G & 0 \\ 0 & \partial_t G & 0 & \partial_t\partial_{t'}G \end{pmatrix}, \tag{5.2}$$

where the row vector, k_m^μ, (μ is the Lorentz index and m is the mode index) multiplying the matrix, is written in the following order: $(k_0^0\,k_0^1\,k_1^0\,k_1^1)$. $G(t,t') = \langle X(t)X(t')\rangle$ where t,t' are points on the boundary of the string world sheet. This could be a disk or the upper half plane. G satisfies the identity $\int dt''\partial_t G(t,t'')\partial_{t''}G(t'',t') = 1$ [17, 18].

The wave functional in this case is

$$\Psi[k_0(t),k_1(t)] = \int \mathcal{D}X(t)e^{-i\int dt[k_0^\mu(t)X^\mu(t)]}\prod_{\mu t}\delta[k_1^\mu(t) - A^\mu(X(t)]\,, \tag{5.3}$$

where $A^\mu = 1/2F^{\mu\nu}X^\nu$. For simplicity we take the two dimensional case, where $\mu = 0,1$. Thus the delta function becomes

$$\prod_t \delta\left[X^1(t) - \frac{2k_1^0(t)}{F}\right]\delta\left[X^0(t) + \frac{2k_0^1(t)}{F}\right]\frac{1}{F^2}\,.$$

Here $F = F^{01} = -F^{10}$.

Doing the the integral gives

$$\Psi[k_0(t),k_1(t)] = e^{\frac{2i}{F}\int dt[k_0^0(t)k_1^1(t)-k_0^1(t)k_1^0(t)]}\prod_t\frac{1}{F^2}\,.$$

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When we substitute this expression in (5.1), we get

$$\int \mathcal{D}k_0(t)\mathcal{D}k_1(t)e^{\frac{1}{2}\int\int dtdt'[\sum_{n,m=0,1}k_m^\mu(t)\mathcal{G}^{\mu\nu}_{F,m,n}(t,t')k_n^\nu(t')]}e^{-i\int dt[k_0^\mu(t)X^\mu(t)+k_1^\mu(t)\partial_t X^\mu(t)]}\,, \tag{5.4}$$

where $\mathcal{G}_F$ is the matrix

$$\begin{pmatrix} G & 0 & \partial_{t'}G & 2i/F \\ 0 & G & -2i/F & \partial_{t'}G \\ \partial_t G & -2i/F & \partial_t\partial_{t'}G & 0 \\ 2i/F & \partial_t G & 0 & \partial_t\partial_{t'}G \end{pmatrix}.$$

The gaussian integral can easily be done and using the identity obeyed by G we get

$$e^{\frac{1}{2}\int\int dtdt'\mathbf{X}^T\mathcal{G}_F^{-1}\mathbf{X}}\mathrm{Det}^{-1/2}[F^{-2}(1+F^{-2})\delta(t-t')] =$$
$$e^{\frac{1}{2}\int\int dtdt'\mathbf{X}^T\mathcal{G}_F^{-1}\mathbf{X}}\mathrm{Det}[F^2(1+F^2)^{-1/2}\delta(t-t')]\,.$$

Here $\mathbf{X} = (X^0,\, X^1,\, \partial_t X^0,\, \partial_t X^1)$.

Combining the $1/F^2$ in the wave functional we get

$$e^{\frac{1}{2}\int\int dtdt'\mathbf{X}^T\mathcal{G}_F^{-1}\mathbf{X}}\mathrm{Det}[1+F^2]^{-1/2}\delta(t-t')\,.$$

Using zeta function regularization (as explained in [18]) the determinant gives $[1+F^2]^{+1/2}$. Thus the final answer is

$$Z[\mathbf{X}] = e^{\frac{1}{2}\int\int dtdt'\mathbf{X}^T\mathcal{G}_F^{-1}\mathbf{X}}[1+F^2]^{+1/2}\,. \tag{5.5}$$

$Z[0]$ is the Born Infeld lagrangian. The precise connection between the sigma model "partition function" and the actual space-time effective action of the fields of the string is not clear to us, although in this particular example they seem to coincide. What we do know is that the β-function gives the equation of motion upto the Zamolodchikov metric prefactor.[4] The equations obtained in the loop variable approach are the equivalent of the β-function. They are a gauge invariant generalization obtained by making the cutoff depend on the world sheet location.

One can obtain the equation for the photon by looking at the coefficient of $\partial_t X^\mu$ in the conformal variation of the above "generating functional with sources X", (5.5). In the loop variable approach this was done by replacing G by $G+\Sigma$ and the coefficient of Σ gave the equation of motion.[5] It is clear here that there are no terms linear in X and the equation is trivial. In order to get a non-trivial equation one has to include a perturbation describing the non-uniformity of F.

[4] It was also shown in [21] that the proper-time equation gives the full equation.

[5] We remind the reader that all the x_n variables used in the loop variable approach have been set to zero, as we are not concerned with gauge invariance related issues.

Thus we change the expression for A into:

$$A^\mu(X) = \frac{1}{2}F^{\mu\nu}X^\nu + \frac{1}{3}\partial_\rho F^{\mu\nu}X^\rho X^\nu \,. \tag{5.6}$$

The delta function in (5.3) changes to

$$\prod_t \delta\left[k_1^\mu(t) - \frac{1}{2}F^{\mu\nu}X^\nu(t) + \frac{1}{3}\partial_\rho F^{\mu\nu}X^\rho(t)X^\nu(t)\right] =$$
$$= \left(1 + \int dt' \frac{1}{3}\partial_\rho F^{\mu\nu}X^\rho(t')X^\nu(t')\frac{\delta}{\delta k_1^\mu(t')}\right)\prod_t \delta\left[k_1^\mu(t) - \frac{1}{2}F^{\mu\nu}X^\nu(t)\right].$$

If we insert this into (5.3) we get for the modified wave functional,

$$\begin{aligned}\Psi[k_0(t), k_1(t)] = &\int \mathcal{D}X(t)e^{-i\int dt[k_0^\mu(t)X^\mu(t)]} \times \\ &\times\left(1 + \int dt' \frac{1}{3}\partial_\rho F^{\mu\nu}X^\rho(t')X^\nu(t')\frac{\delta}{\delta k_1^\mu(t')}\right)\prod_{\mu t}\delta[k_1^\mu(t) - A^\mu(X(t)] \\ = &\left(1 - \int dt'\frac{1}{3}\partial_\rho F^{\mu\nu} i\frac{\delta}{\delta k_0^\rho(t')} i\frac{\delta}{\delta k_0^\nu(t')}\frac{\delta}{\delta k_1^\mu(t')}\right)\Psi[k_0(t), k_1(t)]\,.\end{aligned} \tag{5.7}$$

We insert the modified wave functional into (5.1), integrate by parts on k_1, k_0 and pick the piece proportional to $\dot{X}$:

$$\begin{aligned}&\int\int \mathcal{D}k_0^\mu(t)\mathcal{D}k_1^\mu(t)\int dt'\frac{1}{3}\partial_\rho F^{\mu\nu} i\partial_{t'}X^\mu(t')[\mathcal{G}^{0\rho,0\nu}(t',t') + (\mathcal{G}k)^{0\rho}(\mathcal{G}k)^{0\nu}] \times \\ &\times e^{-i\int dt[k_0^\mu(t)X^\mu(t) + k_1^\mu(t)\partial_t X^\mu(t)]}e^{\frac{1}{2}\int\int dt dt'[\sum_{n,m=0,1}k_m^\mu(t)\mathcal{G}_{m,n}^{\mu\nu}(t,t')k_n^\nu(t')]}\Psi[k_0(t), k_1(t)]\,.\end{aligned} \tag{5.8}$$

Replacing k_0^μ by $i\delta/\delta X^\mu(t')$ we get and k_1^μ by $i\delta/\delta\dot{X}^\mu(t')$, (We let X^α represent $X^\mu, \dot{X}^\mu$)

$$\begin{aligned}&\int dt'\frac{1}{3}\partial_\rho F^{\mu\nu} i\partial_{t'}X^\mu(t')\times \\ &\times\left[\mathcal{G}^{0\rho,0\nu}(t',t') + \int dt''\int dt'''(\mathcal{G}(t',t'')^{0\rho,\alpha} i\frac{\delta}{\delta X^\alpha(t'')}(\mathcal{G}^{0\nu,\beta}(t',t''')i\frac{\delta}{\delta X^\beta(t''')}\right]\times \\ &\times e^{\frac{1}{2}\int\int dt dt'\mathbf{X}^T\mathcal{G}_F^{-1}\mathbf{X}}[1+F^2]^{+1/2} = \\ &= \int dt'\frac{1}{3}\partial_\rho F^{\mu\nu} i\partial_{t'}X^\mu(t')[\mathcal{G}^{0\rho,0\nu}(t',t') - (\mathcal{G}\mathcal{G}_F^{-1}\mathcal{G})^{0\rho,0\nu}(t',t')]Z[\mathbf{X}]\,.\end{aligned} \tag{5.9}$$

One can easily evaluate the expression multiplying $Z[\mathbf{X}]$. It is the matrix

$$\begin{pmatrix} G(t,t')/1+F^2 & \int dt'' iFG(t,t'')\partial_{t''}G(t'',t') \\ -\int dt'' iFG(t,t'')\partial_{t''}G(t'',t') & G(t,t')/1+F^2\end{pmatrix} \tag{5.10}$$

multiplied by $\delta(t'-t)$.

The off diagonal elements are antisymmetric in t, t' and therefore vanish when $t = t'$. Thus we get

$$\delta^{\rho\nu} \int dt \frac{G(t,t')}{1+F^2} \delta(t-t') .$$

When we make a conformal transformation the change in G is Σ so we get a term

$$\frac{1}{3}\partial_\rho F^{\mu\nu} i\partial_{t'} X^\mu(t')\delta^{\rho\nu} \int dt \frac{\Sigma(t,t')}{1+F^2}\delta(t-t') = \frac{1}{3}\partial_\rho F^{\mu\nu} i\partial_{t'} X^\mu(t')\delta^{\rho\nu}\frac{\Sigma(t',t')}{1+F^2} . \tag{5.11}$$

The coefficient of Σ in the above expression is one contribution to the β-function. Variation of $\delta(t-t')$ gives terms proportional to delta function and it's derivatives (times the parameter of conformal transformation). Multiplying G this is singular and introduces further powers of the cutoff. These are equivalent to $(ln\, a)^2$ divergences and do not contribute to the β-function. Thus the coefficient of Σ in (5.11) is the full expression for the β-function.

As explained in [21] one can get the full equation of motion (= β-function $\times$ Zamolodchikov metric) if one uses the proper time equation. But the method here gives only the β-function. That this answer is correct can be seen by consulting [17].

6. Conclusions

In this paper we have given some examples of loop variable calculations. We have tried to make concrete, some of the ideas described in [15, 16]. We have given examples of the wave functional that is ubiquitous in these two papers. We have also worked out in detail the equations of a massive mode interacting with electromagnetism. Most importantly while a formal argument of the gauge invariance was given in [16], there was not a detailed understanding of how the infinite tower of massive modes contribute in the gauge invariance of any one equation. This understanding has now been obtained, as described in section 3.

Before dimensional reduction, all the modes are massless and we are not describing critical string theory - at least not in any recognizable way. Thus dimensional reduction is crucial for making contact with string theory. This was described in a general way in [16]. Here we have worked out all the details in this particular example. We have also shown how the truncation of fields necessary for making contact with string theory works.

Finally, by reproducing the Born-Infeld equations, we make contact with the open string sigma model (or boundary conformal field theory) approach. This was useful in explaining the ideas of [15, 16] in more conventional terms.

It is tempting to speculate that the higher dimensional massless theory is some more "symmetric" phase of string theory. The idea that the interactions emerge naturally by broadening the string to a band is reminiscent of membranes. All this points to M-theory. But we do not see any way of making a connection.

As another interesting test one could try to reproduce the results on the tachyon that have been obtained in the literature using other techniques such as String Field Theory or Background Independent String Field Theory [23]–[29].

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